Dendrites in Nature and in Computer

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1. Introduction

The World is full of “webby”, multibranch objects with tree-like structure, named dendrites, from the Greek word *dendron*: tree. Typical examples are everywhere: connexions of a neural cell, lightnings in the atmosphere, etc. We shall see many more examples. The figures below show the pattern formed by the growth of bacteriae on a planar support, the figure of Hele-Shaw generated by the injection of gaz (or a liquid) into a layer of viscous liquid, and finally, the Lichtenberg figure – an electric discharge inside a dielectric.

Fig. 1. Fear...

Fig. 2. Some examples of natural dendrites

(Currently we have better apparatus than Georg Christoph Lichtenberg (1742–1799). For example, a piece of organic glass is pumped by a beam of electrons from an accelerator, until its explosive discharge. Scientists continue to investigate the properties of those complex sparks, since they provide a plethora of information on the isolation properties of various materials, which is very important in engineering. However, the figures are so nice, that the production of Lichtenberg figures in different frames and shapes became a profitable souvenir business...)

The central picture on the previous page may represent also a section of a piece of coral, or a deposit of metal condensation out of vapour. Many crystals grow showing such patterns, and several young chemists played at home with a little chemical "garden": an aquarium is filled with an aqueous solution of water glass (sodium silicate). Then, we put in some small crystals of metallic sulphates or nitrates: copper, iron, manganese, cobalt, etc., of various colours.

The salts dissolve, and they react with the water glass, producing semipermeable, insoluble membranes of metal silicates. The osmosis makes the water diffuse into the layer between the crystal and the membrane, and this layer expands, until it bursts. Then some salt solution is liberated, and seals immediately the hole, forming new membrane. Since the membrane breaks usually in places where it is fresh and thin, the process produces "fingers" or "tongues", similar to corals.

Of course, it would be preposterous to try to find common *physical mechanisms* responsible for the structure of our blood vessel system, the structure of a neuron, and the affluent system of a big river, but several really distant physical or biological phenomena share some common mathematics. Usually this mathematics is complicated, and some common features need computer simulation in order to have some insight.

2. Dendrites and diffusion; a computer experiment

The following experiment which demonstrates a dendritic aggregation is implemented as a very simple program which can be coded by high-school pupils. A *finite space* represented as a zero-one (2- or 3-dimensional) array, is filled with zeros, only near the center we put one 1 – an immobile "nucleus", a small grain in a vessel filled with liquid. At some small distance of the nucleus we put another grain, not "physically", but we just memorize its position. This second grain may displace itself, and it does so randomly, as in the figure below, at the left. This simulates the Brownian motion, and is implemented as a "drunk's walk": the moving particle makes one step, collides with something, forgets everything, and continues its walk in a random direction...

When this particle eventually touches the nucleus, its position becomes adjacent to that of the immobile grain. Then, it stops and becomes an element of the growing aggregate, the value of the array element becomes 1. The following particles, which start outside the region of the aggregate, undergo the same treatment. Let us repeat this drunk's walk some thousands times. What will be the shape of the growing compound?

The reader who has never seen the solution shown below, could suspect that the nucleus grows as an irregular, but compact blob. But no, we clearly see the appearance of filamentary structures, of dendrites.

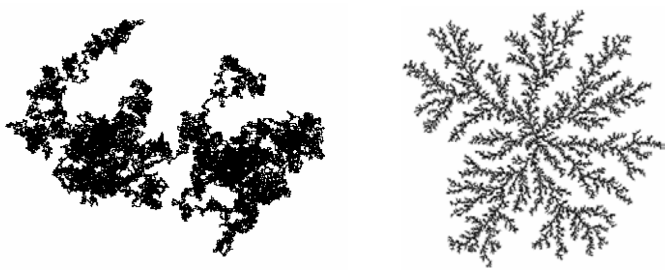


Fig. 3. A sample of a random "diffusion" path, and the result of the aggregation of diffusing particles

Why does this happen? It is *much* easier for the moving grain to collide with the external "corona" of the existing structure, than to pass through narrow channels near the center of the cluster. One can estimate that about 75% of the cluster, forming its central part, does not evolve any more, while the dendrite growth is mainly peripheral. The evolution is quite unstable. If a new particle makes a little "bump" on an existing branch, it will increase the probability that new particles stop at this bump, so it will grow faster than its neighbourhood.

In principle that is all. We can make the array more complicated, so that different zones within the "web" indicate the age of the local particles, and this may be coded as colour. We can choose different initial positions of particles, or restrict its movements to the vicinity of the nucleus; instead of waiting until it comes back, we destroy it, and start with a nearer one.

This will accelerate our simulation. Another way of making it faster is to generate steps bigger than 1 pixel, e.g., the smallest distance between the grain and the cluster, and random direction, with the angle within $[0, 2\pi)$. This is a classical "computer cheating", but algorithmically correct. We lose only the relation between the real time, and the simulated one, since the speed of diffusion has no physical meaning anymore. Plenty of other optimisations are possible, some of them are based on the analysis of some extended neighbourhood of the grain, while in Nature particles are blind, they "feel" the collision only when it takes place.

3. But why this similarity of forms?

Why this experiment is interesting? Well, since the dendrites are everywhere, we can put forward a hypothesis that the *abstract* mechanisms of their generation are – at least partly – universal. If our simulation works well we may be able to understand better their character. We **know** that the topology of several geometric objects are often invariant and universal.

In particular, we can see why the "chemical garden" dendrites have some formal affinities with our clustering process. The diffusion aggregate grows by glueing new particles from outside, while the chemical dendrite from inside, but in both cases the appearance of ***new elements***, new "fingers" increases the chance that the growth will continue at their position. The rich becomes richer, there is no "social justice" in this phenomenon...

On the other hand, the sparks (lightnings, Lichtenberg figures, etc.) are regions of ionized, conducting gas, extending from electrodes. We know that the electric potential of an electrode is constant, but the field intensity depends very strongly on the surface curvature, it is stronger near edges and corners. There, it becomes easier to ionize new volumes of gas, and to extend the region occupied by the spark.

We might say a few words about the theory. Several fields (such as the electric field, or the field of velocities of an uncompressible liquid) fulfil the Laplace differential equation. In two dimensions its form is $(\partial^2/\partial x^2 + \partial^2/\partial y^2)V = 0$. We could prove that if for a given point (x, y) in the space outside the cluster we sum the values of the electric field at the points on the space boundary, from which we get to the given point by following *random paths*, the result will give us a numerical, approximate solution of the Laplace equation! Such a random way of solving this equation is effectively used in many problems in physics and engineering. Random numbers and functions are often useful for solving completely deterministic problems.

A similar mechanism may be applied to the Hele-Shaw figures. We have to analyze the dependence of the liquid surface tension which tries to confine the introduced droplet of liquid, on the local curvature of its surface. Again this is unstable. The analysis of the growth of bacteriae may be left as an exercice. It is sufficient to take into account the fact that the bacteriae must get some food, and if they are too numerous, they poison the environment, making the reproduction more difficult. Corals (or ordinary trees) which also grow in a dendritic way, are too complicated to discuss here, although, as for the bacteriae, the fact that the food (or light) is outside, also plays its role.

It might be useful to look at a counter-example. A forest fire behaves differently. Despite the fact that there is more oxygen outside, and it comes from outside rather than from above (convection), the form of the burning region is blobby rather than dendritic, since the air is in constant motion, and a heated-up tree, well inside the burning zone catches fire much easier.

4. Fractal dimension of dendrites

The mechanism of DLA (*DiffusionLimited Aggregation*), described and simulated already long ago, has been analyzed in details quite recently, in 1981, by T.A.

Witten, and L.M. Sander. We cannot discuss here many interesting details, but we can say a few words on the *topological dimension* of the generated cluster. Our simulation is powerful enough to measure it. But what is this dimension? We know that we live in a 3-dimensional space. A "normal" material object of a given density possesses a mass, which – by the definition of density – is proportional to its volume, and thus to l^3, where l is some characteristic linear parameter of the object, e.g. the radius of a sphere, if the object is more or less spherical.

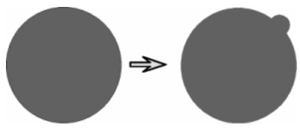
In 2 dimensions the surface replaces the volume. A compact blob will have the dimension 2. A thin thread (line, path) which would be created if its constituent particles positioned themselves one behind the other, without drastic changing of direction, nor self-intersections, would have the dimension 1. The characteristic attribute would be the *length* of the object as a function of the number of particles.

While simulating DLA, we might measure the dependence of the radius – maximal or average – of the aggregate, on the number of particles forming it. It turns out empirically that in the limit of large number of particles, the dependence is $N = \text{const}\cdot r^{1.7}$. Such a non-integer dimension is called *fractal*. We observe a very curious phenomenon, this non-integer exponent is relatively invariant, it has the same value for many distinct physical systems. The DLA process in 3 dimensions, which is as easy to simulate as the 2D version, but much, much slower, gives us the fractal dimension approximately equal to 2.5.

Of course, scientists use much more elaborate models. For example a moving particle may, with a certain probability, refuse to attach itself to the cluster, and continue to move. This gives dendrites more compact and "furry", like a flake of soot. The dependence of the fractal exponent and the viscosity of the particles gives us interesting information on the structure of such matter.

5. Model of Hastings and Levitov

We shall discuss now a not well known (and rarely taught) model of dendritic development, based on the theory of analytic functions. This is a complex and involved model, and the reader is not expected to follow all the details. But the foundations are straightforward to explain. In 1998 M.B. Hastings and L.S. Levitov noticed that the solution of the 2D Laplace equation use the theory of functions on the complex plane, and the s.c. conformal transformation. By itself, this was known for many years.

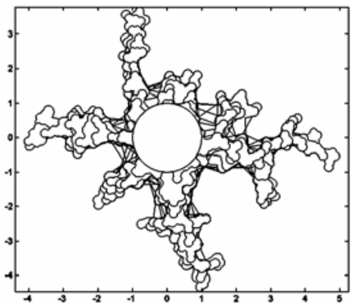


They reasoned: let's analyze the transformation $f_{r,\phi}(z)$ of a unit disk on a complex plane z, i.e., the region which fulfils the constraint $|z| \le 1$, into a disk with an extruded "finger", as shown in this figure.

The parameters (r,ϕ) specify the position and the orientation of the extrusion. The angle ϕ will be randomly drawn from the interval (0 – 2π), and r depends on the stage of the generation process. The function f will permit the construction of **much** more complicated transformation which makes a dendrite out of a unit disk. If we knew this incredibly complicated function $D(z)$ which specifies the dendrite containing, say, N particles (local extrusions), then $D(f(z))$ will describe the dendrite with one more element. So, we take a computer, we begin with $D_0(z)$ which defines the disk of radius R, and we iterate. The function f is not *very* complicated, but it is not too pleasant either, for example $f_{r,\phi}(z) = e^{i\phi} f_r(z) e^{-i\phi}$, where

$$f_r(z) = \sqrt{z}\left\{(1+r)\frac{1+z}{2z}\left[1+z+z\sqrt{1+\frac{1}{z^2}-\frac{2}{z}\frac{1-r}{1+r}}\right]-1\right\}^{1/2}$$

(Additionally, during the iterations we must rescale r, in order to preserve correct proportions.) We can easily show that already the second or the third iteration create functions so complicated that almost impossible to write down analytically. However, we can do it numerically, storing in the memory of the computer an array which represents the function contour. Each iteration produces a new geometric layer. A few initial stages are shown below. Here r does not decrease too fast, so we obtain a finger-like structure, not filaments.



We shall not discuss the programme which creates this figure, it is not very simple. Our aim was to demonstrate that the theory of complex functions, sometimes presented to students in an abstract way, finds some unexpected applications in physics. We show also that computers enable us to manipulate *functions* which are so complicated, that impossible to put down on paper, since they are the outcome of several thousands of iterations.

Our text contains also some philosophical ideas, since it argues for the *unity of Nature*. In many, apparently very distinct contexts we see phenomena of unstable geometrical evolution. We introduce the Laplace equation (often being the conclusion of some conservation laws and the continuity), we operate upon a system with a short memory (Markov systems), and we build some mathematical models for those systems. Often, if we succeed, we realize that those models are similar. So, the behaviour of all those systems is similar, despite dramatic structural differences between them.

Sometimes a tree-like development is not relevant to a single object, but it is distributed among several generations, and many, many years. From a goal-oriented perspective sometimes used to analyze evolution, a neuron "wants" to communicate with as many neighbours as possible. The blood vessels "want" to irrigate the biggest possible volume of a body in an economical way. So, the instabilities do not concern individuals, but genetics: speciments which possessed those more branched apparata, were more clever or more robust, so that they could transmit their genes more efficiently.

The generation of dendrites is a simple, but characteristic, archetypical example of *morphogenesis* which is the source of the richness of forms in the Universe.

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